Affine Schemes

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1 Motivation

At the very beginning of any course in algebraic geometry, one starts off with the definition of an affine variety. Recall that we then defined projective varieties, and noted that they were somehow composed of several "affine patches".

Example 1. Consider the elliptic curve $V \subset \mathbb{P}^2$ given by the homogeneous equation $Y^2Z = X^3 + XZ^2$.

That is, $V = \{[X, Y, Z] \in \mathbb{P}^2 : Y^2Z = X^3 + XZ^2\}$.

We recover the "affine patches" of which V is composed by setting one of the variables to 1. That is, we get the three affine varieties $V_1, V_2, V_3 \subset \mathbb{A}^2$ defined by:

$$\begin{aligned} V_1 &= \{(x, y) \in \mathbb{A}^2 : y^2 = x^3 + x\} \\ V_2 &= \{(x, z) \in \mathbb{A}^2 : z = x^3 + xz^2\} \\ V_3 &= \{(y, z) \in \mathbb{A}^2 : y^2z = z^2 + 1\}. \end{aligned}$$

From our notion of projective variety, we were then able to talk about "varieties" without specifically meaning affine or projective ones by constructing an abstract definition of "variety".

In exactly the same way, we are going to start our study of schemes with the definition of an affine scheme, and then construct projective schemes by "patching together" affine schemes. Thence we will be able to define schemes in general.

2 Affine Varieties and Affine Schemes

As Dan described in his lecture last week, there is a great correspondence between Algebra and Geometry. Through the Nullstellensatz, we associate the following objects (assuming from now on that k is an algebraically closed field):

Algebra	$\longleftrightarrow$	Geometry
$k[x_1, \dots, x_n]$	$\longleftrightarrow$	$\mathbb{A}^n$
radical ideals	$\longleftrightarrow$	affine algebraic sets
prime ideals	$\longleftrightarrow$	affine varieties
maximal ideals	$\longleftrightarrow$	points in $\mathbb{A}^n$

We are interested in affine varieties and how to generalise them to affine schemes, so let's recall the explicit way in which prime ideals correspond to affine varieties.

An affine algebraic set $V \subset \mathbb{A}^n$ is irreducible (i.e. it is an affine variety) if and only if its ideal $V(I) \subset k[x_1, \ldots, x_n]$ is a prime ideal.

In that case, we can form the *affine coordinate ring*, denoted $k[V]$ and defined as the quotient ring $k[V] := k[x_1, \ldots, x_n]/V(I)$. Because $I(V)$ is prime, $k[V]$ is an integral domain, and so we can define the *function field* $k(V)$ to be the field of fractions of $k[V]$.

Let's take a moment to see what we've done here. We have associated a ring $k[V]$ with an affine variety V. What properties (as a ring) does $k[V]$ have? It is:

- finitely generated (this is the content of the Hilbert Basis Theorem),
- nilpotent-free (i.e. there are no nonzero elements $f \in k[V]$ which satisfy $f^r = 0$ for some integer r), and
- an algebra over k.

A natural question to ask is: "Given a finitely-generated, nilpotent-free algebra R over an algebraically closed field k, does there exist an affine variety V such that $R = k[V]$?"

YES!

In fact, there is a bijective correspondence between such rings R and affine varieties V.

We want to know if there is a way to generalise this correspondence. The question is: "Given an *arbitrary* commutative ring R, can we construct geometric objects in a similar way?" The answer, of course, is yes, and the geometric objects we obtain are called *affine schemes*.

3 Spectra, Sheaves and the Zariski Topology

Given an arbitrary commutative ring R, we will define the *affine scheme* associated to it to consist of three things:

1. The set $Spec(R)$, the set of all prime ideals of R,
2. A topology on that set called the *Zariski topology*, and
3. A *sheaf* of regular functions, called the *structure sheaf.*

Let's look at the example of $\mathbb{A}^2$, and consider it as an affine scheme.

Example 2. The affine variety $\mathbb{A}^2$ corresponds with the ring $R = \mathbb{C}[V] = \mathbb{C}[x, y]$. The spectrum of R is as follows:

$$Spec(R) = \{(x - a, y - b) : a, b \in \mathbb{C}\} \cup \{(f) : f(x, y) \text{ is irreduciblel}\} \cup \{(0)\}.$$

Thus, we (somewhat unintuitively) consider $\mathbb{A}^2$ as consisting of its points (a, b) (corresponding to the maximal ideals $(x - a, y - b)$), AND its irreducible subvarieties given by the equations $f(x, y) = 0$ (i.e. the irreducible curves lying in the plane). This is a bit unsettling at first, but it turns out to be very convenient.

The Zariski topology on $\mathbb{A}^2$ is the usual Zariski topology, where the closed sets are the zero sets of polynomials. Dan defined the Zariski topology last week: for each subset $S \subset R$, the closed set which corresponds to S is:

$$V(S) = \{P \in Spec(R) : S \subset P\}.$$

It is easy to check that each closed set does indeed correspond to an algebraic set in $\mathbb{A}^2$.

In this example, the structure sheaf describes the regular functions which exist on $\mathbb{A}^2$. The structure sheaf, as we shall see, assigns to each open set U of $\mathbb{A}^2$ a ring of functions $U \longrightarrow \mathbb{C}$ which are regular on U.

The next few lectures will try to understand these sheaves of regular functions.

As motivation, let us consider what we mean by "regular functions on the whole of $Spec(R)$ for a general ring R.

Note that each element $f \in R$, defines a "function" on $Spec(R)$ in the following sense. Let $P \in Spec(R)$. Then R/P is an integral domain, and so we can form the field of fractions, called the *residue field* and denoted $\kappa(P)$. We define $f(P)$ to be f, as considered as an element of $\kappa(P)$.

Let us end with an example.

Example 3. Let $R = \mathbb{Z}$, so that $Spec(R) = \{(p) : p \text{ is a prime number}\}$.

The element $15 \in R$ acts as a "function" on the element $(7) \in Spec(R)$ by $15((7)) = 15$ as an element of $\mathbb{Z}/(p)$, i.e. $15((7)) = 8 \bmod 7$.

References

[1] D. Eisenbud and J. Harris, *The geometry of schemes*, Springer-Verlag, New York, 2000.